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Viscous boundary layers in flows through a domain with permeable boundary

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Dedicated to the memory of Victor Yudovich (1934–2006)

Abstract

The effect of small viscosity on nearly inviscid flows of an incompressible fluid through a given domain with permeable boundary is studied. The Vishik–Lyusternik method is applied to construct a boundary layer asymptotic at the outlet in the limit of vanishing viscosity. Mathematical problems with both consistent and inconsistent initial and boundary conditions at the outlet are considered. It is shown that in the former case, the viscosity leads to a boundary layer only at the outlet. In the latter case, in the leading term of the expansion there is a boundary layer at the outlet and there is no boundary layer at the inlet, but in higher order terms another boundary layer appears at the inlet. To verify the validity of the expansion, a number of simple examples are presented. The examples demonstrate that asymptotic solutions are in quite good agreement with exact or numerical solutions.

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1. Introduction

In this paper, we study the effect of small viscosity on nearly inviscid flows of an incompressible fluid through a given domain. In contrast with the standard situation where the boundary of the flow domain is impermeable, we consider flows in which the fluid can flow into and out of the domain through its boundary. A typical example is a flow in a pipe of finite length when the ends of the pipe represent the inlet and the outlet of the flow domain. Note that, in general, the permeable boundary does not necessarily correspond to any physical boundary (made, say, of a porous material), it may be any fixed surface at which we can control or merely measure the appropriate flow parameters. An interesting general question that arises in this context is: what boundary data do we need to control (or just to know) in order to control (or to describe) the temporal evolution of the flow in the domain provided that the initial velocity is known.

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Inviscid flows of this type had been studied by many authors (see, e.g, [1–8]). In contrast with the conventional case of impermeable boundary, there are many different well-posed initial boundary value problems for the Euler equations in a domain with permeable boundary. Usually, the normal velocity is prescribed by a boundary condition for the Euler equations. In the case of nonzero normal velocity at the boundary, additional boundary conditions are necessary. They must be posed at the inlet (i.e. on the part of the boundary where the fluid enters the domain), where, in addition to the normal velocity, one needs to specify the tangent component of vorticity (see [1–4]), or the tangent component of velocity (see [3–5]), or some other quantities (see [6,7] and references therein). Note that Yudovich [1,2] had proved that the two-dimensional problem with the vorticity specified at the inlet always has a unique global (in time) solution. For all other initial-boundary value problems with nonzero normal velocity, only local existence and uniqueness theorems are known even in the two-dimensional case, and there are no examples of collapse.

Here we study the effect of small viscosity on nearly inviscid flows through a given domain. This paper may be viewed as an extension of the earlier work of Yudovich [9] who constructed an asymptotic boundary layer at the outlet for a two-dimensional incompressible flow in an infinite strip bounded by two straight lines, one of which serves as the inlet and the other as the outlet. Alekseenko [10] and Temam and Wong [11,12] considered similar problems and obtained a number of results about the convergence of solutions of the Navier–Stokes equations to corresponding solutions of the Euler equations. Much earlier a related problem about a boundary layer on a rigid wall with suction had been studied by numerous authors (see, e.g., [13,14] and references therein). All early works had Prandtl's boundary layer equations as a starting point of the analysis. This implies that the normal (suction) velocity through the boundary is small, of order of $Re^{-1/2}$ (where Re is the Reynolds number). In this paper, we are interested in flows at high Reynolds numbers in which the normal velocity is of order of unity. A particular problem of steady boundary layer with strong suction (the suction velocity goes to infinity) had been studied by Watson [15] and some others (see [14] and references therein). Watson [15] had obtained a solution of Prandtl's boundary layer equations in the form of a series in inverse powers of the suction velocity. Although Watson's result seems to be correct and is in agreement (at least in the leading order term) with what is described below, his procedure involves somewhat contradictory assumptions (he considers the limit of large suction velocity in the Prandtl equations that are obtained under the assumption that this velocity is small).

In this paper, we construct an asymptotic expansion of solutions of the Navier–Stokes equations which describe flows through an arbitrary three-dimensional domain in the limit of high Reynolds numbers (vanishing viscosity). In order to focus on special features of a flow through a domain with permeable boundary, we restrict our analysis to the case where no part of the boundary is a rigid impermeable wall (i.e. the entire boundary is a union of the inlet and the outlet). To construct the asymptotic expansion, we follow the ideas of [9] and employ the Vishik–Lyusternik method (see, e.g., [16,17]). The method had been used to study viscous boundary layer at a fixed impermeable boundary by Chudov [18]. In the leading term of the expansion, the flow is described by the Euler equations everywhere in the flow domain except for a thin boundary layer at the outlet. The equations of the boundary layer turn out to be linear and can be solved analytically in the general case. It is interesting that the initial boundary value problem for the boundary layer equations that arises in the leading order is over-determined, but nevertheless has a unique solution provided that the initial condition for the velocity is consistent with the boundary condition for the velocity at the outlet. It is shown that, in principle, the asymptotic expansion can be computed up to terms of arbitrary order in the inverse Reynolds number provided that certain additional consistency requirements on initial and boundary data at the outlet are satisfied.

We also consider a mathematical problem with inconsistent initial and boundary conditions at the outlet which can be used to describe real fluid flows produced by a sudden change in the tangent velocity at the outlet. In this case, the asymptotic expansion constructed describes fast relaxation of an inconsistency between initial and boundary conditions for the tangent velocity at the outlet. It turns out that in higher order terms of the expansion another boundary layer may appear at the inlet in contrast with the case of consistent initial and boundary conditions where there is no boundary layer at the inlet in terms of all orders.

The plan of the paper is the following. In Section 2, we describe the method of constructing the asymptotic expansion for a two-dimensional incompressible flow in an infinite strip bounded by two straight lines, one of which serves as the inlet and the other as the outlet. In Section 3, we construct the asymptotic expansion in the general case of a flow through an arbitrary three-dimensional domain. Section 4 deals with the case of inconsistent initial and boundary conditions at the outlet. In Section 5, we discuss the results.

2. Boundary layer at the outlet in a simple geometry

In order to avoid treatment of boundary layers on rigid impermeable boundaries, we restrict our analysis to the situation where no part of the boundary of the flow domain is impermeable, i.e. the boundary is a union of the inlet and the outlet. In order to introduce the Vishik–Lyusternik method (that is employed throughout the paper) in the simplest possible way, we first discuss the particular problem that was treated earlier by Yudovich [9] and Temam and Wang [11,12].

2.1. Formulation of the problem

Consider the two-dimensional Navier–Stokes equations

$$u_t + (u\partial_x + v\partial_y)u = -p_x + \nu\nabla^2 u, \tag{2.1}$$

$$v_t + (u\partial_x + v\partial_y)v = -p_y + \nu\nabla^2 v, \tag{2.2}$$

$$u_x + v_y = 0 \tag{2.3}$$

in the domain $\mathcal{D}$ (see Fig. 1)

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < \infty,\ 0 < y < L\},$$

which is a strip between the lines $y = 0$ and $y = L$ in the xy plane. In Eqs. (2.1)–(2.3), u and v are the Cartesian components of the velocity, and p is the pressure divided by the density. All quantities are made dimensionless using some characteristic scales V^* for velocity, L^* for length, L^*/V^* for time and V^{*2} for pressure divided by density, so that ν is the dimensionless viscosity (the inverse Reynolds number): $\nu = \nu^*/V^*L^*$ where ν^* is the kinematic viscosity of the fluid. Boundary conditions at $y = 0$ and $y = L$ are

$$u(x, y, t)|_{y=0} = U(x, t), \tag{2.4}$$

$$v(x, y, t)|_{y=0} = V(x, t), \tag{2.5}$$

and

$$u(x, y, t)|_{y=L} = U_1(x, t), \tag{2.6}$$

$$v(x, y, t)|_{y=L} = V_1(x, t). \tag{2.7}$$

We assume that

$$V(x, t) < 0, \quad \text{and} \quad V_1(x, t) < 0 \quad \text{for all } x. \tag{2.8}$$

So, the fluid flows into the domain through the plane $y = L$ (the inlet) and leaves it through the plane $y = 0$ (the outlet). (In general, one may prescribe other boundary conditions at the inlet and outlet, for example, the vorticity $\omega = v_x - u_y$ may be given instead of conditions (2.4) and (2.6) for tangent velocity.)

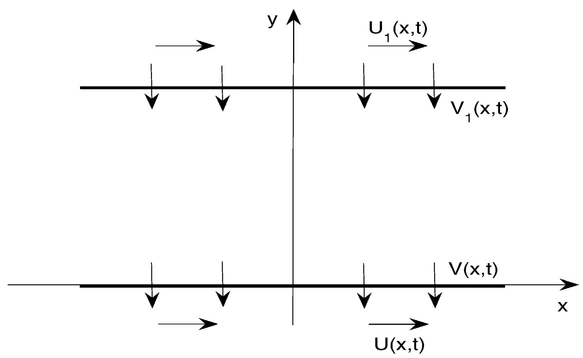

Fig. 1. Sketch of the flow domain.

Initial conditions:

$$u(x, y, t)|_{t=0} = U_0(x, y), \tag{2.9}$$

$$v(x, y, t)|_{t=0} = V_0(x, y). \tag{2.10}$$

Functions $U(x,t)$, $V(x,t)$, $U_1(x,t)$, $V_1(x,t)$, $U_0(x,y)$, $V_0(x,y)$ are either periodic in variable x with the period L_x or decay as $|x| \to \infty$, and in what follows we look for solutions of the problem (2.1)–(2.10) which have the same property.

We assume also that the initial and boundary conditions are consistent, i.e.

$$U_0(x, y)|_{y=0} = U(x, t)|_{t=0}, \qquad V_0(x, y)|_{y=0} = V(x, t)|_{t=0}, \tag{2.11}$$

$$U_0(x, y)|_{y=L} = U_1(x, t)|_{t=0}, \qquad V_0(x, y)|_{y=L} = V_1(x, t)|_{t=0}. \tag{2.12}$$

We are interested in studying the solutions of the initial boundary value problem (2.1)–(2.10) in the limit $\nu \to 0$. In what follows, we employ the Vishik–Lyusternik method to construct the formal asymptotic solution of the problem.

2.2. Asymptotic expansion

We seek the solution of (2.1)–(2.10) in the form

$$u = u^i(x, y, t) + u^b(x, s, t), \tag{2.13}$$

$$v = v^i(x, y, t) + \nu v^b(x, s, t), \tag{2.14}$$

$$p = p^i(x, y, t) + \frac{1}{\nu} p^b(x, s, t), \tag{2.15}$$

where $s = y/\nu$. Functions u^i, v^i, p^i represent a regular expansion of the solution in power series in ν (an outer solution), and u^b, v^b, p^b correspond to a boundary layer correction (an inner solution) to this regular expansion. The scaling factors, ν for v^b and ν^{-1} for p^b are chosen in order to preserve the form of the incompressibility equation and keep the pressure term in the leading order equation for u^b. This makes subsequent analysis easier. In this particular problem, the choice of scaling factors does not play a crucial role. For example, one can get the same result by assuming that both scaling factors are equal to unity.

Let

$$\mathbf{v}^i = \mathbf{v}_0^i + \nu \mathbf{v}_1^i + \nu^2 \mathbf{v}_2^i + \cdots, \qquad p^i = p_0^i + \nu p_1^i + \nu^2 p_2^i + \cdots, \tag{2.16}$$

where $\mathbf{v}^i = (u^i, v^i)$, $\mathbf{v}_k^i = (u_k^i, v_k^i)$ $(k = 0, 1, 2, \ldots)$. The successive approximations $\mathbf{v}_k^i$, p_k^i $(k = 0, 1, 2, \ldots)$ satisfy the equations:

$$\partial_t \mathbf{v}_0^i + (\mathbf{v}_0^i \cdot \nabla)\mathbf{v}_0^i = -\nabla p_0^i, \tag{2.17}$$

$$\nabla \cdot \mathbf{v}_0^i = 0, \tag{2.18}$$

and

$$\partial_t \mathbf{v}_k^i + (\mathbf{v}_0^i \cdot \nabla)\mathbf{v}_k^i + (\mathbf{v}_k^i \cdot \nabla)\mathbf{v}_0^i = -\nabla p_k^i + \mathbf{F}_k, \tag{2.19}$$

$$\nabla \cdot \mathbf{v}_k^i = 0 \tag{2.20}$$

for $k = 1, 2, \ldots$. Here

$$\mathbf{F}_1 = \nabla^2 \mathbf{v}_0^i, \qquad \mathbf{F}_k = \nabla^2 \mathbf{v}_{k-1}^i - \sum_{n=1}^{k-1} (\mathbf{v}_n^i \cdot \nabla)\mathbf{v}_{k-n}^i \quad (k = 2, 3, \ldots).$$

The leading term of the outer expansion satisfies the Euler equations (2.17), (2.18). Higher order terms are solutions of the linearised Euler equations with external forces which are determined by lower order approximations. Since the viscous terms are absent in Eqs. (2.17) and (2.19), the outer solution cannot satisfy all the boundary conditions. It is known that the Euler equation has a unique solution at least locally in time if both components of the velocity are prescribed at the inlet and only normal component is given at the outlet (see, e.g., [6]). We require that, in the leading order, the outer solution satisfies the following boundary conditions

$$v_0^i|_{y=0} = V(x,t), \quad v_0^i|_{y=L} = V_1(x,t), \quad u_0^i|_{y=L} = U_1(x,t). \tag{2.21}$$

Thus, the condition for the tangent velocity at the outlet (2.4) is not necessarily satisfied. We need the boundary layer part of the solution to satisfy this condition. The initial conditions for u_0^i and v_0^i are

$$u_0^i|_{t=0} = U_0(x,y), \tag{2.22}$$

$$v_0^i|_{t=0} = V_0(x,y). \tag{2.23}$$

2.3. Boundary layer

We assume that

$$u^b = u_0^b + \nu u_1^b + \cdots, \quad v^b = v_0^b + \nu v_1^b + \cdots, \quad p^b = p_0^b + \nu p_1^b + \cdots. \tag{2.24}$$

We substitute (2.13)–(2.16), (2.24) into Eqs. (2.1)–(2.3) and take into account that u_k^i, v_k^i, p_k^i $(k = 0, 1, \ldots)$ satisfy the equations (2.17)–(2.20). Then we make the change of variables $y = \nu s$, expand every function of νs in Taylor's series at $\nu = 0$ and, finally, collect terms of the equal powers in ν. In the leading order, we obtain

$$V(x,t)\partial_s u_0^b = -\partial_x p_0^b + \partial_s^2 u_0^b, \tag{2.25}$$

$$0 = -\partial_s p_0^b, \tag{2.26}$$

$$\partial_x u_0^b + \partial_s v_0^b = 0. \tag{2.27}$$

Here we used the fact that in the boundary layer,

$$v_0^i(x,y,t) = v_0^i(x,\nu s,t) = v_0^i(x,0,t) + \mathrm{O}(\nu) = V(x,t) + \mathrm{O}(\nu).$$

We require that

$$u_0^b \to 0, \quad v_0^b \to 0, \quad p_0^b \to 0 \quad \text{as } s \to \infty. \tag{2.28}$$

It follows from (2.26) that $p_0^b(x,s,t) = f(x,t)$ for some $f(x,t)$. Then, the conditions (2.28) imply that $p_0^b(x,s,t) \equiv 0$. It follows from (2.25) that

$$V(x,t)\partial_s u_0^b = \partial_s^2 u_0^b. \tag{2.29}$$

Boundary condition for u_0^b at $s = 0$ is given by

$$u_0^b(x,s,t)|_{s=0} = h_0(x,t) \equiv U(x,t) - u_0^i(x,y,t)|_{y=0}. \tag{2.30}$$

The initial condition is

$$u_0^b(x,s,t)|_{t=0} = 0. \tag{2.31}$$

Note that the initial boundary value problem (2.28)–(2.31) is an over-determined problem, because the equation (2.29) does not contain the derivative with respect to time. Generically, problems like this have no solutions. Fortunately, the problem (2.28)–(2.31) does have a unique solution. Indeed, the solution of Eq. (2.29) subject to the boundary conditions (2.28) and (2.30) is given by

$$u_0^b(x,s,t) = h_0(x,t)e^{V(x,t)s} \tag{2.32}$$

(recall that $V(x,t) < 0$ for all x). Then we observe that if the initial and boundary conditions for $u(x,y,t)$ in the original problem are consistent, i.e. satisfy the first relation (2.11), then

$$h_0(x,t)|_{t=0} = 0, \tag{2.33}$$

so that the solution, given by (2.32), satisfies the initial condition (2.31). Thus, the remarkable fact is that (2.32) represents the unique solution of the over-determined initial boundary value problem (2.28)–(2.31) provided that the initial and boundary conditions in the original problem are consistent.

The main difference between the boundary layer equations (2.25)–(2.27) and Prandtl's boundary layer at an impermeable rigid wall is that the equations are linear and can be easily solved. Another important difference is that the characteristic thickness of the boundary layer at the outlet is $O(\nu)$ rather than $O(\sqrt{\nu})$. As will be seen later, we can write down a universal solution of the boundary layer equations in arbitrary geometry, and this may lead to a mathematically rigorous justification of the formal asymptotic solution obtained here.

Eq. (2.27) is used to find v_0^b:

$$v_0^b(x,s,t) = \frac{\partial}{\partial x}\int_s^\infty u_0^b(x,s',t)\,ds' = -\frac{\partial}{\partial x}\left[\frac{h_0(x,t)}{V(x,t)}e^{V(x,t)s}\right]. \tag{2.34}$$

Here we have chosen the constant of integration so as to satisfy the boundary condition at infinity (2.28). Note that, in general, v_0^b does not satisfy the boundary condition $v_0^b(x,s,t)|_{s=0} = 0$. Fortunately, in view of (2.14), the correction to v_0^i is νv_0^b. Therefore,

$$v_0^b(x,s,t)|_{s=0} = -\frac{\partial}{\partial x}\left[\frac{h_0(x,t)}{V(x,t)}\right] \tag{2.35}$$

gives us the boundary condition for the next approximation of the outer solution:

$$v_1^i(x,y,t)|_{y=0} = -v_0^b(x,s,t)|_{s=0} = \frac{\partial}{\partial x}\left[\frac{h_0(x,t)}{V(x,t)}\right]. \tag{2.36}$$

2.4. Higher order terms

Higher order terms in the inner expansion are determined from the equations

$$\begin{aligned} &V(x,t)\partial_s u_k^b = -\partial_x p_k^b + \partial_s^2 u_k^b + a_k(x,s,t),\\ &\partial_s p_k^b = b_k(x,s,t),\\ &\partial_x u_k^b + \partial_s v_k^b = 0 \quad (k = 1,2,\ldots), \end{aligned}$$

where functions $a_k(x,s,t)$, $b_k(x,s,t)$ are defined in terms of lower order solutions u_m^i, v_m^i, p_m^i, u_m^b, v_m^b, p_m^b ($m = 0,1,\ldots,k-1$). For example,

$$\begin{aligned} &b_1 = 0,\\ &a_1 = -\partial_t u_0^b - u_0^i(x,0,t)\partial_x u_0^b - u_0^b\partial_x u_0^i(x,0,t) - u_0^b\partial_x u_0^b - \big(v_0^b + v_1^i(x,0,t) + s\partial_y v_0^i(x,0,t)\big)\partial_s u_0^b. \end{aligned} \tag{2.37}$$

To clarify the procedure of constructing higher-order terms, let us consider the first-order terms of the asymptotic expansion in detail. We have

$$V(x,t)\partial_s u_1^b = -\partial_x p_1^b + \partial_s^2 u_1^b + a_1(x,s,t), \tag{2.38}$$

$$\partial_s p_1^b = 0, \tag{2.39}$$

$$\partial_x u_1^b + \partial_s v_1^b = 0. \tag{2.40}$$

Here we have used the fact that $b_1(x,s,t) \equiv 0$. Functions u_1^b, v_1^b and p_1^b must satisfy the conditions

$$u_1^b \to 0, \quad v_1^b \to 0, \quad p_1^b \to 0 \quad \text{as } s \to \infty. \tag{2.41}$$

As before, the condition of decay at infinity for p_1^b and Eq. (2.39) imply that $p_1^b \equiv 0$, so that we obtain the separate equation for u_1^b:

$$V(x,t)\partial_s u_1^b = \partial_s^2 u_1^b + a_1(x,s,t). \tag{2.42}$$

Boundary condition for u_1^b at $s = 0$ is given by

$$u_1^b(x,s,t)|_{s=0} = h_1(x,t) \equiv -u_1^i(x,y,t)|_{y=0}. \tag{2.43}$$

Here $u_1^i(x,y,t)$ is the first-order term of the outer expansion which satisfies the equations (cf. Eqs. (2.19), (2.20))

$$\partial_t u_1^i + (u_0^i \partial_x + v_0^i \partial_y) u_1^i + (u_1^i \partial_x + v_1^i \partial_y) u_0^i = -\partial_x p_1^i + \nabla^2 u_0^i, \tag{2.44}$$

$$\partial_t v_1^i + (u_0^i \partial_x + v_0^i \partial_y) v_1^i + (u_1^i \partial_x + v_1^i \partial_y) v_0^i = -\partial_y p_1^i + \nabla^2 v_0^i, \tag{2.45}$$

$$\partial_x u_1^i + \partial_y v_1^i = 0, \tag{2.46}$$

subject to the boundary condition (2.36), the boundary conditions

$$u_1^i|_{y=L} = 0, \qquad v_1^i|_{y=L} = 0, \tag{2.47}$$

as well as the initial conditions

$$u_1^i|_{t=0} = 0, \qquad v_1^i|_{t=0} = 0. \tag{2.48}$$

Note that since $h_0(x,t)|_{t=0} = 0$, the boundary condition (2.36) is consistent with the initial conditions (2.48), i.e. $v_1^i|_{y=0,\,t=0} = 0$, and therefore there are no principal difficulties in solving the 'outer' problem (2.44)–(2.48), (2.36).

The initial conditions (2.48) and Eq. (2.43) imply that

$$u_1^b|_{s=0,\,t=0} = h_1(x,0) = 0. \tag{2.49}$$

The initial condition for the boundary layer part of the expansion is

$$u_1^b(x,s,t)|_{t=0} = 0. \tag{2.50}$$

So, again we have an over-determined problem for u_1^b, because Eq. (2.42) does not contain the derivative with respect to time. However, below we show that it has a unique solution provided that a certain additional consistency condition (for the initial and boundary conditions in the original initial boundary value problem for the Navier–Stokes equations) is satisfied.

First, we observe that, in view of (2.49), the initial condition (2.50) is consistent with the boundary condition (2.43). As was shown above, the consistency of the initial and boundary condition at $y = 0$ in the original problem is sufficient to guarantee the existence of a unique solution of the leading order boundary layer equations satisfying initial condition (2.31). However, this is not true for the first-order boundary layer equation (2.42). The reason for this is that Eq. (2.42) is nonhomogeneous. Indeed, the solution of Eq. (2.42) that satisfies boundary conditions (2.41), (2.43) can be written in the form

$$u_1^b(x,s,t) = h_1(x,t) e^{V(x,t)s} + e^{V(x,t)s} \int_0^s e^{-V(x,t)s'} \int_{s'}^{\infty} a_1(x,s'',t)\, ds''\, ds'. \tag{2.51}$$

It follows from (2.49) that

$$u_1^b(x,s,t)|_{t=0} = e^{V(x,0)s} \int_0^s e^{-V(x,0)s'} \int_{s'}^{\infty} a_1(x,s'',0)\, ds''\, ds'. \tag{2.52}$$

Evidently, the solution given by Eq. (2.51) satisfies the initial condition (2.50) only if $a_1(x,s,0) = 0$. On inspecting the expression for $a_1(x,s,t)$, we find that, in view of (2.32), (2.33) and (2.34),

$$a_1(x,s,0) = -\partial_t u_0^b(x,s,t)|_{t=0} = -e^{V(x,0)s} \partial_t h_0(x,t)|_{t=0}. \tag{2.53}$$

Thus, to ensure that problem (2.41)–(2.43), (2.50) has a unique solution (given by (2.51)) we must require that $\partial_t h_0(x,t)|_{t=0} = 0$ or, equivalently,

$$\partial_t U(x,t)|_{t=0} = \partial_t u_0^i(x,0,t)|_{t=0}. \tag{2.54}$$

This represents an additional consistency requirement on the initial and boundary conditions in the original problem. To see this, we need to express the right hand side of Eq. (2.54) in terms of the initial and boundary data. On passing to the limit $t \to 0$ in Eq. (2.17) and using the initial conditions (2.9) and (2.10), we obtain

$$\partial_t \mathbf{v}_0^i|_{t=0} = -(\mathbf{V}_0 \cdot \nabla)\mathbf{V}_0 - \nabla p_0^i|_{t=0}. \tag{2.55}$$

Here $\mathbf{V}_0 = (U_0(x,y), V_0(x,y))$ is the initial velocity (cf. Eqs. (2.9) and (2.10)). The initial pressure $P_0(x,y) = p_0^i(x,y,0)$ can be found from (2.17) and (2.18) by taking the divergence of (2.17) and then passing to the limit $t \to 0$. As a result, $P_0(x,y)$ is the solution of the problem:

$$\Delta P_0 = -\nabla \cdot (\mathbf{V}_0 \cdot \nabla)\mathbf{V}_0,$$
$$\partial_y P_0|_{y=0} = -(\mathbf{V}_0 \cdot \nabla) V_0|_{y=0}, \qquad \partial_y P_0|_{y=L} = -(\mathbf{V}_0 \cdot \nabla) V_0|_{y=L}. \tag{2.56}$$

Now if we pass to the limit $y \to 0$ in Eq. (2.55) and substitute the result in Eq. (2.54), we obtain the required consistency condition

$$\partial_t U(x,t)|_{t=0} = -\big[(\mathbf{V}_0 \cdot \nabla)U_0 + \partial_x P_0\big]|_{y=0}. \tag{2.57}$$

Thus, *in order to obtain the uniquely determined first-order term of the asymptotic expansion, we must impose the additional consistency condition* (*given by* (2.57)) *on the initial and boundary data in the original initial boundary value problem for the Navier–Stokes equation*.

It can be shown that the second-order term in the expansion requires yet another additional consistency condition, and so on. In principle, it is possible to compute as many terms of the asymptotic expansion as necessary. In Section 3, we will show how to modify our asymptotic expansion to deal with the problems with inconsistent initial and boundary conditions.

Remark 2.1. The asymptotic expansion constructed above is applicable to steady or time-periodic flows. In these cases, the analysis is much simpler because there are no initial conditions and, therefore, no consistency conditions are required. An example of a time-periodic flow in an annulus between two concentric circular cylinders will be given in the end of Section 3.

Remark 2.2. The appearance of new consistency conditions in construction of higher-order terms of the asymptotic expansion is not entirely unexpected. Similar conditions emerge in inviscid flows through a domain with permeable boundary [1,2,9]. It turns out that the consistency of initial and boundary conditions at the inlet guarantees that the inviscid solution is continuous, while to ensure that the first derivatives of the velocity with respect to spatial variables are continuous, an additional consistency condition for initial and boundary data at the inlet must be satisfied. Continuity of the second derivatives requires yet another consistency condition, and so on.

2.5. Example 1

Consider the two-dimensional Navier–Stokes equations (2.1)–(2.3) in the same domain $\mathcal{D}$ with the following boundary conditions

$$v|_{y=0} = v|_{y=L} = -1, \tag{2.58}$$
$$u|_{y=0} = 4\sin t, \tag{2.59}$$
$$u|_{y=L} = 1. \tag{2.60}$$

We assume that $u(x,y,t)$, $v(x,y,t)$ and $p(x,y,t)$ are bounded functions for all $(x,y) \in \mathcal{D}$ and that $u(x,y,t)$ and $v(x,y,t)$ satisfy the initial conditions

$$v|_{t=0} = -1, \tag{2.61}$$
$$u|_{t=0} = 1 - (1-y)^4. \tag{2.62}$$

The initial and boundary conditions are chosen so as to satisfy the primary consistency conditions (2.11)–(2.12) and, in addition, the first-order consistency condition (2.57) for the tangent velocity at the outlet.

The solution of the problem (2.1)–(2.3), (2.58)–(2.62) has the form

$$v = -1, \quad u = u(y,t), \quad p = 0,$$

where $u(y,t)$ satisfies the equation

$$\partial_t u - \partial_y u = \nu \partial_y^2 u \tag{2.63}$$

and the boundary and initial conditions (2.59), (2.60) and (2.62).

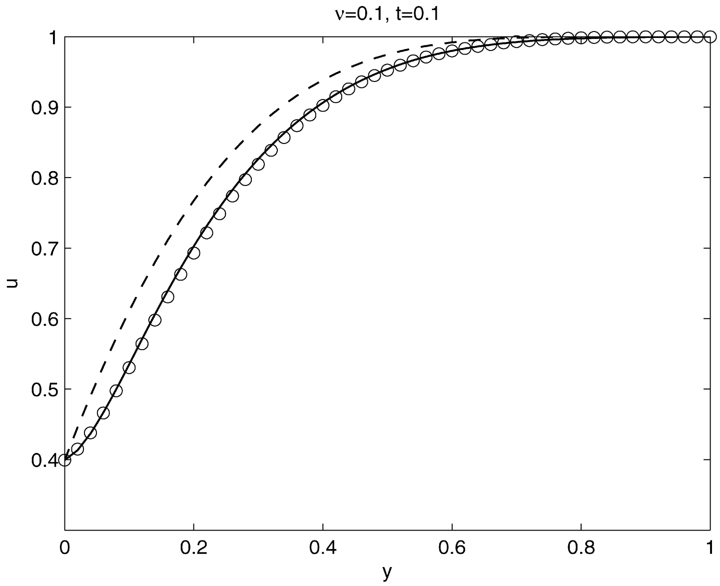

Fig. 2. Leading order and first-order asymptotic solutions $u(y,t)|_{t=0.15}$ for $\nu = 0.1$. Dashed line corresponds to the leading order approximation $u_0 = u_0^i + u_0^b$, solid line – to the numerical solution, and circles represent the first-order asymptotic solution $u_0 + \nu u_1$ where $u_1 = u_1^i + u_1^b$.

The leading term of the outer asymptotic expansion is given by

$$u_0^i = \begin{cases} 1-(1-y-t)^4, & 0 \leqslant y < 1-t, \\ 1, & 1-t \leqslant y \leqslant 1 \end{cases} \quad \text{for } 0 \leqslant t \leqslant 1,$$
$$u_0^i = 1 \quad \text{for } t > 1. \tag{2.64}$$

The leading term of the boundary layer part of the expansion is given by

$$u_0^b = \begin{cases} [4\sin t - 1 + (1-t)^4]e^{-y/\nu}, & 0 \leqslant t \leqslant 1, \\ (4\sin t - 1)e^{-y/\nu}, & t > 1. \end{cases}$$

It can be shown that the first-order terms of the expansion are

$$u_1^i = \begin{cases} -12(1-y-t)^2, & 0 \leqslant y < 1-t, \\ 0, & 1-t \leqslant y \leqslant 1 \end{cases} \quad \text{for } 0 \leqslant t \leqslant 1,$$
$$u_1^i = 0 \quad \text{for } t > 1, \tag{2.65}$$

and

$$u_1^b = \begin{cases} \{12(1-t)^2 - 4[\cos t - (1-t)^3]\}\frac{y}{\nu}e^{-y/\nu}, & 0 \leqslant t \leqslant 1, \\ -4\cos t\,\frac{y}{\nu}e^{-y/\nu}, & t > 1. \end{cases}$$

An example of the leading order approximation $u_0 = u_0^i + u_0^b$, the first-order approximation $u_0 + \nu u_1$, where $u_1 = u_1^i + u_1^b$, and the numerical solution obtained using the standard MATLAB PDE solver are shown in Fig. 2. One can see that the first-order asymptotic solution almost coincides with the numerical solution.

3. Boundary layer at the outlet in general case

The aim of this section is to show briefly how the theory of Section 2 can be generalised to the three-dimensional flows in arbitrary domains with smooth boundaries.

3.1. Formulation of the problem

We consider the Navier–Stokes equations, written in the form

$$\partial_t \mathbf{v} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla Q - \nu \nabla \times (\nabla \times \mathbf{v}), \tag{3.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3.2}$$

in an arbitrary three-dimensional domain $\mathcal{D}$ with sufficiently smooth boundary $\partial\mathcal{D}$. Here $\mathbf{v} = (u, v, w)$ is the velocity, $Q = p + \mathbf{v}^2/2$ is the Bernoulli function and p is the pressure. We do not want to deal with boundary layers at rigid walls, so we assume that no part of $\partial\mathcal{D}$ is a rigid wall and $\partial\mathcal{D}$ consists of the inlet Γ^+ and the outlet Γ^- only:

$$\partial\mathcal{D} = \Gamma^+ \cup \Gamma^-.$$

We assume that all three components of the velocity are given on $\partial\mathcal{D}$:

$$\mathbf{v}|_{\partial\mathcal{D}} = \mathbf{V}(x, y, x, t). \tag{3.3}$$

By the definition of the inlet and the outlet,

$$\begin{aligned} &\mathbf{v} \cdot \mathbf{n}|_{\Gamma^+} = \mathbf{V}(x, y, x, t) \cdot \mathbf{n}|_{\Gamma^+} = \mathbf{V}^+(x, y, x, t) \cdot \mathbf{n} < 0, \\ &\mathbf{v} \cdot \mathbf{n}|_{\Gamma^-} = \mathbf{V}(x, y, x, t) \cdot \mathbf{n}|_{\Gamma^-} = \mathbf{V}^-(x, y, x, t) \cdot \mathbf{n} > 0, \end{aligned} \tag{3.4}$$

where $\mathbf{n}$ is the unit outward normal on $\partial\mathcal{D}$. The initial condition

$$\mathbf{v}|_{t=0} = \mathbf{V}_0(x, y, x) \tag{3.5}$$

is assumed to be consistent with the boundary condition (3.3).

We look for an asymptotic representation of a solution to Eqs. (3.1)–(3.3), (3.5) as $\nu \to 0$ in the form

$$\mathbf{v} = \mathbf{v}^i + \mathbf{v}^b, \qquad Q = Q^i + \frac{1}{\nu} Q^b, \tag{3.6}$$

where $\mathbf{v}^i$ and Q^i represent an outer part of the asymptotic expansion, $\mathbf{v}^b$ and Q^b correspond to a boundary layer part which is nonzero only in a thin layer near the outlet Γ^-.

3.2. Outer expansion

We seek the outer part of the asymptotic solution in the form of the power series in ν:

$$\begin{aligned} &\mathbf{v}^i = \mathbf{v}_0^i + \nu \mathbf{v}_1^i + \nu^2 \mathbf{v}_2^i + \cdots, \\ &Q^i = Q_0^i + \nu Q_1^i + \nu^2 Q_2^i + \cdots. \end{aligned} \tag{3.7}$$

We insert these into Eqs. (3.1), (3.2) and collect the terms of equal order in ν. This yields the following equations:

$$\partial_t \mathbf{v}_0^i - \mathbf{v}_0^i \times (\nabla \times \mathbf{v}_0^i) = -\nabla Q_0^i, \tag{3.8}$$

$$\nabla \cdot \mathbf{v}_0^i = 0, \tag{3.9}$$

and

$$\partial_t \mathbf{v}_k^i - \mathbf{v}_0^i \times (\nabla \times \mathbf{v}_k^i) - \mathbf{v}_0^i \times (\nabla \times \mathbf{v}_k^i) = -\nabla Q_k^i + \mathbf{F}_k, \tag{3.10}$$

$$\nabla \cdot \mathbf{v}_k^i = 0 \tag{3.11}$$

for $k = 1, 2, \ldots$. Here

$$\begin{aligned} &\mathbf{F}_1 = -\nabla \times (\nabla \times \mathbf{v}_0^i), \\ &\mathbf{F}_k = -\nabla \times (\nabla \times \mathbf{v}_{k-1}^i) + \sum_{n=1}^{k-1} \big(\mathbf{v}_n^i \times (\nabla \times \mathbf{v}_{k-n}^i)\big) \end{aligned} \tag{3.12}$$

for $k = 2, 3, \ldots$. The leading term of the outer expansion satisfies the Euler equations (3.8), (3.9). Higher order terms are solutions of the linearised Euler equations with external forces which are determined by lower order approximations. As discussed in Section 2, the outer solution cannot satisfy all the boundary conditions. Recalling that the Euler equation has a unique solution at least locally in time if all three components of the velocity are prescribed at the inlet and only normal component is given at the outlet (see, e.g., [6]), we seek an outer solution which, in the leading order, satisfies the following boundary conditions

$$\mathbf{v}_0^i|_{\Gamma^+} = \mathbf{V}^+, \tag{3.13}$$

$$\mathbf{v}_0^i \cdot \mathbf{n}|_{\Gamma^-} = \mathbf{V}^- \cdot \mathbf{n}. \tag{3.14}$$

Thus, the condition for the tangent velocity at the outlet is not necessarily satisfied. As in Section 2, we need to construct the boundary layer part of the solution to satisfy this condition.

The boundary conditions and the initial conditions for higher order terms $\mathbf{v}_k^i$ ($k = 1, 2, \ldots$) in outer expansion have the form

$$\mathbf{v}_k^i|_{\Gamma^+} = 0, \tag{3.15}$$

$$\mathbf{v}_k^i \cdot \mathbf{n}|_{\Gamma^-} = g_k(\mathbf{x}), \tag{3.16}$$

$$\mathbf{v}_k^i \cdot \mathbf{n}|_{t=0} = 0, \tag{3.17}$$

where function $g_k(\mathbf{x})$ is determined from the boundary layer part of the asymptotic expansion which is considered below.

3.3. Boundary layer

In a neighbourhood of Γ^- we introduce orthogonal curvilinear coordinates q_1, q_2, q_3 in such a way that Γ^- is the coordinate surface $q_3 = q_3^0$. This can be done in many ways. For example, within the boundary layer whose thickness is less than the minimal radius of curvature of the surface Γ^- we can define q_3 as the distance from the point $\mathbf{x}$ to Γ^- (note that in this case, $q_3^0 = 0$). Let u^b, v^b, w^b be the components of the boundary layer correction $\mathbf{v}^b$ to the outer solution that correspond to coordinates q_1, q_2, q_3, respectively. Then we introduce the boundary layer variable (inner variable) s as

$$s = \frac{q_3 - q_3^0}{\nu}. \tag{3.18}$$

We look for $u^b(q_1, q_2, s, t)$, $v^b(q_1, q_2, s, t)$, $w^b(q_1, q_2, s, t)$, $Q^b(q_1, q_2, s, t)$ in the form

$$\begin{aligned}
u^b &= u_0^b + \nu u_1^b + \nu^2 u_2^b + \cdots, \\
v^b &= v_0^b + \nu v_1^b + \nu^2 v_2^b + \cdots, \\
w^b &= \nu(w_0^b + \nu w_1^b + \nu^2 w_2^b + \cdots), \\
Q^b &= Q_0^b + \nu Q_1^b + \nu^2 Q_2^b + \cdots.
\end{aligned} \tag{3.19}$$

To derive the boundary layer equations, we need first to rewrite the original Navier–Stokes equations (3.1), (3.2) in the curvilinear coordinates q_1, q_2, q_3. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the unit vectors parallel to coordinate curves, and let H_1, H_2, H_3 be the Lame coefficients:

$$H_i^2 = \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial q_i} \right)^2. \tag{3.20}$$

In curvilinear coordinates q_1, q_2, q_3, formulas for $\nabla\phi$, $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ given by (see, e.g., Appendix 2 in [19])

$$\nabla\phi = \frac{1}{H_1}\frac{\partial\phi}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial\phi}{\partial q_2}\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial\phi}{\partial q_3}\mathbf{e}_3, \tag{3.21}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q_1}(H_2 H_3 u) + \frac{\partial}{\partial q_2}(H_3 H_1 v) + \frac{\partial}{\partial q_3}(H_1 H_2 w)\right], \tag{3.22}$$

$$\nabla\times\mathbf{v}=\frac{1}{H_2H_3}\left[\frac{\partial}{\partial q_2}(H_3w)-\frac{\partial}{\partial q_3}(H_2v)\right]\mathbf{e}_1+\frac{1}{H_1H_3}\left[\frac{\partial}{\partial q_3}(H_1u)-\frac{\partial}{\partial q_1}(H_3w)\right]\mathbf{e}_2$$
$$+\frac{1}{H_1H_2}\left[\frac{\partial}{\partial q_1}(H_2v)-\frac{\partial}{\partial q_2}(H_1u)\right]\mathbf{e}_3. \tag{3.23}$$

Substituting these into Eqs. (3.1) and (3.2), we obtain the required form of the Navier–Stokes equations. To derive the boundary layer equations, we need to substitute (3.6), (3.7), (3.19) into the Navier–Stokes equations and take into account that $\mathbf{v}^i_k$, Q^i_k ($k=0,1,\ldots$) satisfy Eqs. (3.8)–(3.11). Further, we make the change of variables (3.18), expand every function of νs in Taylor's series at $\nu=0$ and, finally, collect terms of the equal powers in ν.

Before doing this, it is convenient to carry out the boundary layer expansion of differential operations grad, div and curl. Making the change of variables

$$q_3=q_3^0+\nu s,\quad \frac{\partial}{\partial q_3}=\frac{1}{\nu}\frac{\partial}{\partial s},\quad w=\nu\hat{w} \tag{3.24}$$

in (3.21)–(3.23), we obtain

$$\nabla\phi=\frac{1}{\nu}\frac{1}{H_3^0}\frac{\partial\phi}{\partial s}\mathbf{e}_3+\mathrm{O}(1), \tag{3.25}$$

$$\nabla\cdot\mathbf{v}=\frac{1}{H_3^0}\frac{\partial\hat{w}}{\partial s}+\frac{1}{H_1^0H_2^0H_3^0}\left[\frac{\partial}{\partial q_1}(H_2^0H_3^0u)+\frac{\partial}{\partial q_2}(H_3^0H_1^0v)\right]+\mathrm{O}(\nu), \tag{3.26}$$

$$\nabla\times\mathbf{v}=\frac{1}{\nu}\frac{1}{H_3^0}\left[-\frac{\partial v}{\partial s}\mathbf{e}_1+\frac{\partial u}{\partial s}\mathbf{e}_2\right]+\mathrm{O}(1). \tag{3.27}$$

Here we have used the formula

$$H_i(q_1,q_2,q_3^0+\nu s)=H_i^0(q_1,q_2,q_3^0)+\mathrm{O}(\nu),\quad i=1,2,3.$$

Similarly, the leading term of the boundary layer expansion of $\nabla\times(\nabla\times\mathbf{v})$ is given by

$$\nabla\times(\nabla\times\mathbf{v})=\frac{1}{\nu^2}\frac{1}{(H_3^0)^2}\left[-\frac{\partial^2u}{\partial s^2}\mathbf{e}_1-\frac{\partial^2v}{\partial s^2}\mathbf{e}_2+\mathrm{O}(\nu)\right]. \tag{3.28}$$

With the help of formulas (3.25)–(3.28), it is not difficult to derive the leading order equations of the boundary layer:

$$\frac{W^-(q_1,q_2,t)}{H_3^0}\frac{\partial u_0^b}{\partial s}=-\frac{1}{H_3^0}\frac{\partial Q_0^b}{\partial q_1}+\frac{1}{(H_3^0)^2}\frac{\partial^2u_0^b}{\partial s^2}, \tag{3.29}$$

$$\frac{W^-(q_1,q_2,t)}{H_3^0}\frac{\partial v_0^b}{\partial s}=-\frac{1}{H_3^0}\frac{\partial Q_0^b}{\partial q_2}+\frac{1}{(H_3^0)^2}\frac{\partial^2u_0^v}{\partial s^2}, \tag{3.30}$$

$$\frac{1}{H_3^0}\frac{\partial Q_0^b}{\partial s}=0, \tag{3.31}$$

$$\frac{1}{H_3^0}\frac{\partial w_0^b}{\partial s}+\frac{1}{H_1^0H_2^0H_3^0}\left[\frac{\partial}{\partial q_1}(H_2^0H_3^0u)+\frac{\partial}{\partial q_2}(H_3^0H_1^0v)\right]=0. \tag{3.32}$$

Here we have used the expansion

$$w_0^i(q_1,q_2,q_3^0+\nu s,t)=w_0^i(q_1,q_2,0,t)+\mathrm{O}(\nu)=W^-(q_1,q_2,t)+\mathrm{O}(\nu),$$

where w_0^i and $W^-(q_1,q_2,t)$ are the components of the vectors $\mathbf{v}_0^i(\mathbf{x},t)$ and $\mathbf{V}^-(\mathbf{x},t)$ respectively that correspond to the coordinate q_3 in the curvilinear coordinate system q_1,q_2,q_3. Note that $\mathbf{V}^-(x,y,x,t)\cdot\mathbf{n}=-W^-(q_1,q_2,t)>0$. Therefore,

$$W^-(q_1,q_2,t)<0.$$

We require that

$$u_0^b \to 0, \quad v_0^b \to 0, \quad w_0^b \to 0, \quad Q_0^b \to 0 \quad \text{as } s \to \infty. \tag{3.33}$$

From (3.31), we deduce that $Q_0^b(q_1, q_2, s, t) = f(q_1, q_2, t)$ for some $f(q_1, q_2, t)$. Then, the conditions (3.33) imply that $Q_0^b(q_1, q_2, s, t) \equiv 0$. This means that the pressure in the boundary layer is determined by the external pressure. Thus, we have

$$W^-(q_1, q_2, t)\frac{\partial u_0^b}{\partial s} = \frac{1}{H_3^0}\frac{\partial^2 u_0^b}{\partial s^2}, \tag{3.34}$$

$$W^-(q_1, q_2, t)\frac{\partial v_0^b}{\partial s} = \frac{1}{H_3^0}\frac{\partial^2 v_0^b}{\partial s^2}. \tag{3.35}$$

Boundary conditions for u_0^b and v_0^b at $s = 0$ are given by

$$u_0^b(q_1, q_2, s, t)|_{s=0} = h_1(x, t) \equiv U^-(q_1, q_2, t) - u_0^i(q_1, q_2, q_3, t)\big|_{q_3=q_3^0}, \tag{3.36}$$

$$v_0^b(q_1, q_2, s, t)|_{s=0} = h_2(x, t) \equiv V^-(q_1, q_2, t) - v_0^i(q_1, q_2, q_3, t)\big|_{q_3=q_3^0}. \tag{3.37}$$

Here u_0^i, v_0^i and $U^-(q_1, q_2, t)$, $V^-(q_1, q_2, t)$ are the components of the vectors $\mathbf{v}_0^i(\mathbf{x}, t)$ and $\mathbf{V}^-(\mathbf{x}, t)$ respectively that correspond to the coordinates q_1, q_2 in the curvilinear coordinate system q_1, q_2, q_3. Since the outer solution satisfies the correct initial conditions, the initial conditions for u_0^b, v_0^b are

$$u_0^b(q_1, q_2, s, t)|_{t=0} = 0, \qquad v_0^b(q_1, q_2, s, t)|_{t=0} = 0. \tag{3.38}$$

The solution of the boundary value problem (3.33)–(3.37) is

$$u_0^b(q_1, q_2, s, t) = h_1(q_1, q_2, t)e^{W^-(q_1, q_2, t)s},$$
$$v_0^b(q_1, q_2, s, t) = h_2(q_1, q_2, t)e^{W^-(q_1, q_2, t)s}. \tag{3.39}$$

We assume that the initial and boundary conditions for $\mathbf{v}(\mathbf{x}, t)$ in the original problem are consistent, i.e. satisfy the relation

$$\mathbf{V}^+(\mathbf{x}, t)|_{t=0} = \mathbf{V}_0(\mathbf{x})|_{\Gamma^+}, \qquad \mathbf{V}^-(\mathbf{x}, t)|_{t=0} = \mathbf{V}_0(\mathbf{x})|_{\Gamma^-}. \tag{3.40}$$

Then, the solution given by Eq. (3.39) satisfies the initial condition (3.38). Thus, we have found two tangent components of the velocity.

The normal component w_0^b is determined from the incompressibility condition (3.32). Integrating Eq. (3.32), we find that

$$w_0^b(q_1, q_2, s, t) = \frac{1}{H_1^0 H_2^0}\int_s^\infty \left[\frac{\partial}{\partial q_1}\big(H_2^0 H_3^0 u(q_1, q_2, s', t)\big) + \frac{\partial}{\partial q_2}\big(H_3^0 H_1^0 v(q_1, q_2, s', t)\big)\right] ds'. \tag{3.41}$$

Here the constant of integration is chosen so as to satisfy the boundary condition at infinity. Note that, in general, $w_0^b(q_1, q_2, s, t)$ does not satisfy the boundary condition at $s = 0$:

$$w_0^b(q_1, q_2, s, t)|_{s=0} \neq 0.$$

w_0^b, however, gives us a correction of order $O(\nu)$. So, the quantity $w_0^b(q_1, q_2, s, t)|_{s=0}$ serves as the boundary value for the first-order terms in the outer expansion.

Higher-order terms of the asymptotic expansion can be obtained in the same way as it was done in Section 2.

3.4. Example 2

Consider a two-dimensional rotationally symmetric flow in the annulus $R_1 < \sqrt{x^2 + y^2} < R_2$ (see Fig. 3). The two-dimensional Navier–Stokes equations, written in polar coordinates (r, θ), are

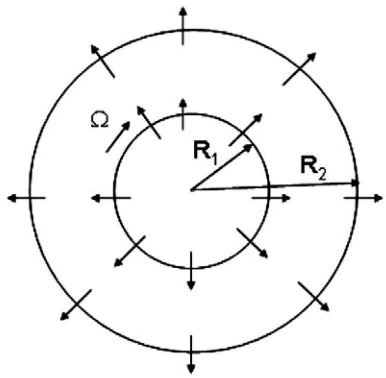

Fig. 3. Flow domain in Example 2.

$$u_t + uu_r + \frac{v}{r}u_\theta - \frac{v^2}{r} = -p_r + \nu\left(\Delta u - \frac{u}{r^2} - \frac{2}{r^2}v_\theta\right), \tag{3.42}$$

$$v_t + uv_r + \frac{v}{r}v_\theta + \frac{uv}{r} = -\frac{1}{r}p_\theta + \nu\left(\Delta v - \frac{v}{r^2} + \frac{2}{r^2}u_\theta\right), \tag{3.43}$$

$$(ru)_r + v_\theta = 0, \tag{3.44}$$

where u and v are the radial and azimuthal components of the velocity.

We seek a rotationally symmetric (i.e. independent of θ) solution of (3.42)–(3.44) which satisfies the boundary conditions

$$u|_{r=R_1} = \frac{q}{R_1}, \qquad v|_{r=R_1} = \Omega(t)R_1, \tag{3.45}$$

$$u|_{r=R_2} = \frac{q}{R_2}, \qquad v|_{r=R_2} = 0. \tag{3.46}$$

Here q is a constant, $2\pi q$ being the total flux of the fluid through the circle $r = R_1$ (or $r = R_2$). We assume that $q > 0$, so that the circle $r = R_1$ is the inlet and the circle $r = R_2$ is the outlet.

The assumption that $u = u(r,t)$, $v = v(r,t)$, $p = p(r,t)$, the incompressibility condition (3.44) and the boundary conditions for u imply that

$$u = \frac{q}{r}. \tag{3.47}$$

Substituting (3.47) into (3.43), we obtain the following linear parabolic equation for $v(r,t)$:

$$v_t + \frac{q}{r}\left(v_r + \frac{v}{r}\right) = \nu\left(v_{rr} + \frac{1}{r}v_r - \frac{v}{r^2}\right). \tag{3.48}$$

Substitution of (3.47) into (3.42) yields the formula for $p(r,t)$ in terms of q and $v(r,t)$:

$$p(r,t) = \int\left(\frac{q^2}{r^3} + \frac{v^2(r,t)}{r}\right)dr. \tag{3.49}$$

Thus, the problem of finding a rotationally-symmetric solution of Eqs. (3.42)–(3.44) which satisfies the boundary conditions (3.45)–(3.46) reduces to the problem of solving Eq. (3.48) subject to the boundary conditions

$$v|_{r=R_1} = \Omega(t)R_1, \tag{3.50}$$

$$v|_{r=R_2} = 0. \tag{3.51}$$

Let

$$\Omega(t) = A\sin\lambda t + B\cos\lambda t = \mathrm{Re}(Ce^{i\lambda t}), \tag{3.52}$$

where $C \equiv B - iA$, A and B are constants. We seek a time periodic solution of (3.48), (3.50), (3.51) in the limit of vanishing viscosity: $\nu \to 0$. The outer part of the solution is $v^i = v_0^i + \mathrm{O}(\nu)$, where v_0^i is the solution of the equation

$$v_t + \frac{q}{r}\left(v_r + \frac{v}{r}\right) = 0$$

that satisfies the boundary condition (3.50). This solution is

$$v_0^i(r,t) = \frac{A}{r}\sin\left[\lambda\left(t - \frac{r^2 - R_1^2}{2q}\right)\right] + \frac{B}{r}\cos\left[\lambda\left(t - \frac{r^2 - R_1^2}{2q}\right)\right]$$
$$= \frac{1}{r}\mathrm{Re}\left[Ce^{i\lambda(t - \frac{r^2 - R_1^2}{2q})}\right].$$

Introducing the boundary layer variable near the outlet by the formula

$$s = \frac{R_2 - r}{\nu},$$

we obtain the following equation for $v_0^b(r,t)$ (cf. Eqs. (3.34), (3.35)):

$$-\frac{q}{R_2}\frac{\partial v_0^b}{\partial s} = \frac{\partial^2 v_0^b}{\partial s^2},$$

whose solution, satisfying the boundary conditions

$$v_0^b|_{s=0} = h(t) = -v_0^i(R_2, t), \qquad v_0^b \to 0 \quad \text{as } s \to \infty,$$

is given by (cf. Eq. (3.39)):

$$v_0^b(s,t) = -\frac{1}{R_2}\mathrm{Re}\left[Ce^{i\lambda(t - \frac{R_2^2 - R_1^2}{2q})}\right]e^{-\frac{q}{R_2}s}.$$

Thus, the leading term in the asymptotic expansion is

$$v(r,t) = \frac{1}{r}\mathrm{Re}\left[Ce^{i\lambda(t - \frac{r^2 - R_1^2}{2q})}\right] - \frac{1}{R_2}\mathrm{Re}\left[Ce^{i\lambda(t - \frac{R_2^2 - R_1^2}{2q})}\right]e^{-\frac{q}{R_2}\frac{R_2 - r}{\nu}} + \mathrm{O}(\nu). \tag{3.53}$$

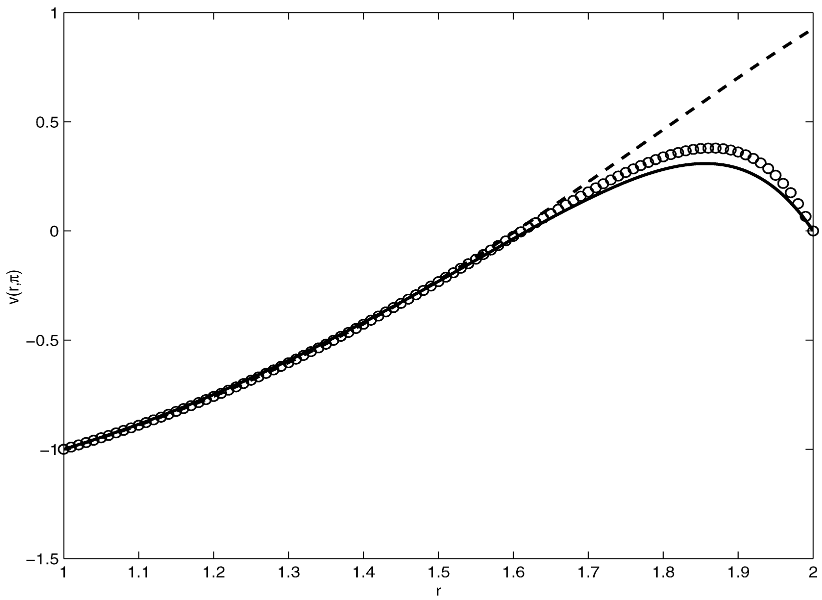

Fig. 4. Inviscid, exact and asymptotic solutions $v(r,t)|_{t=\pi}$ for $\nu = 0.05$, $\lambda = 1$, $A = B = 1$, $R_1 = 1$ and $R_2 = 2$. Dashed line corresponds to the inviscid solution $v_0^i(r,t)|_{t=\pi}$, solid line – to the exact solution (3.54), and the circles represent the asymptotic solution given by Eq. (3.53).

It can be shown that the exact periodic solution of (3.48), (3.50), (3.51) with $\Omega(t)$ given by (3.49) is

$$v(r,t)=\left(\frac{r}{R_1}\right)^{\delta}\mathrm{Re}\left[Ce^{i\lambda t}\frac{I_\delta(\eta r)K_\delta(\eta R_2)-I_\delta(\eta R_2)K_\delta(\eta r)}{I_\delta(\eta R_1)K_\delta(\eta R_2)-I_\delta(\eta R_2)K_\delta(\eta R_1)}\right], \tag{3.54}$$

where

$$\delta=1+\frac{q}{2\nu},\qquad \eta=\sqrt{\frac{\lambda}{\nu}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right),$$

and where $I_\delta(z)$ is the modified Bessel function of the first kind and $K_\delta(z)$ is the modified Bessel function of the second kind.

The exact solution (3.54), the asymptotic solution (3.53) and the inviscid solution v_0^i, shown in Fig. 4, demonstrate quite good agreement between the exact and asymptotic solutions.

4. Inconsistent initial and boundary conditions

The above theory has been developed under the assumption that the initial and boundary conditions are consistent. In particular, the consistency conditions for the tangent components of the velocity at the outlet have a consequence that the boundary layer part of the solution (given by Eq. (2.32) in the two-dimensional case and by Eq. (3.39) in the three-dimensional case) satisfies the required initial conditions (given by Eqs. (2.31) and (3.38)). It is therefore evident that the theory of Sections 2 and 3 does not work if the initial and boundary conditions for the tangent velocity at the outlet are inconsistent. On the other hand, problems with inconsistent initial and boundary conditions are interesting not only from a theoretical perspective but also from a practical viewpoint because they can be used to describe real fluid flows produced by a sudden change in the tangent velocity at the outlet (an example of such flow will be given in Section 4.4).

As was shown in Section 2, even if the consistency conditions (2.11), (2.12) are satisfied, in order to obtain the next (first-order in ν) term in the asymptotic expansion, we have to impose an additional consistency requirement (given by (2.57)) on the initial and boundary conditions in the original problem. So, if we want to solve an initial boundary value problem for the Navier–Stokes equations for which the secondary consistency condition (2.57) is not satisfied, we also must modify our asymptotic expansion. In this section, we will discuss the question of how to generalise the theory of Sections 2 and 3 in the case of inconsistent initial and boundary conditions.

For simplicity of exposition we consider the two-dimensional problem described in Section 2. We assume that the initial condition for the tangent velocity does not agree with the boundary condition at the outlet, i.e. $U(x,t)|_{t=0}\neq U_0(x,y)|_{y=0}$ (the other consistency conditions, for $v(x,y,t)$ at both the inlet and the outlet and for $u(x,y,t)$ at the inlet, are assumed to be satisfied). In this case, $h_0(x,t)|_{t=0}\neq 0$ and, therefore, u_0^b, given by Eq. (2.32), does not satisfy the initial condition (2.31). Physically, it is clear that the initial discontinuity at the boundary will rapidly become smooth due to viscosity. To describe this process, we introduce the fast variable (fast time) $\tau=t/\nu$ and assume that

$$u=u(x,y,t,\tau),\qquad v=v(x,y,t,\tau),\qquad p=\frac{1}{\nu}q(x,y,t,\tau).$$

Inserting these into Eqs. (2.1)–(2.3), we obtain

$$u_\tau+q_x+\nu(u_t+uu_x+vu_y)=\nu^2\nabla^2u, \tag{4.1}$$

$$v_\tau+q_y+\nu(v_t+uv_x+vv_y)=\nu^2\nabla^2v, \tag{4.2}$$

$$u_x+v_y=0. \tag{4.3}$$

Further, we look for an asymptotic expansion in the form

$$u=u^i(x,y,t,\tau)+u^b(x,s,t,\tau), \tag{4.4}$$

$$v=v^i(x,y,t,\tau)+\nu v^b(x,s,t,\tau), \tag{4.5}$$

$$q=q^i(x,y,t,\tau)+q^b(x,s,t,\tau), \tag{4.6}$$

where $s=y/\nu$, as before.

4.1. Outer expansion

To calculate the outer solution u^i, v^i, q^i, we substitute the formulas

$$u^i = u_0^i + \nu u_1^i + \cdots, \quad v^i = v_0^i + \nu v_1^i + \cdots, \quad q^i = q_0^i + \nu q_1^i + \cdots \tag{4.7}$$

into Eqs. (2.1)–(2.3) and collect the terms of equal order in ν. This yields the following equations:

$$\partial_\tau \mathbf{v}_0^i + \nabla q_0^i = 0, \tag{4.8}$$

$$\nabla \cdot \mathbf{v}_0^i = 0, \tag{4.9}$$

$$\partial_\tau \mathbf{v}_1^i + \nabla q_1^i = -\big[\partial_t \mathbf{v}_0^i + (\mathbf{v}_0^i \cdot \nabla)\mathbf{v}_0^i\big], \tag{4.10}$$

$$\nabla \cdot \mathbf{v}_1^i = 0, \tag{4.11}$$

$$\partial_\tau \mathbf{v}_2^i + \nabla q_2^i = -\big[\partial_t \mathbf{v}_1^i + (\mathbf{v}_0^i \cdot \nabla)\mathbf{v}_1^i + (\mathbf{v}_1^i \cdot \nabla)\mathbf{v}_0^i\big] + \Delta \mathbf{v}_0^i, \tag{4.12}$$

$$\nabla \cdot \mathbf{v}_2^i = 0, \quad \text{etc.} \tag{4.13}$$

Boundary conditions for $\mathbf{v}_0^i$ are given by Eqs. (2.21). Eqs. (4.8) and (4.9) imply that

$$\Delta q_0^i = 0, \tag{4.14}$$

and differentiation of the boundary conditions for the normal velocity with respect to τ yields

$$\partial_y q_0^i|_{y=0} = 0, \qquad \partial_y q_0^i|_{y=L} = 0. \tag{4.15}$$

(As before we assume that ∇q_0^i is either periodic in x or decays as $|x| \to \infty$.) The only solution of Eqs. (4.14) and (4.15) satisfying the periodicity condition in x (or the decay condition at infinity) is a constant solution: $q_0^i \equiv$ const. It follows that $\partial_\tau \mathbf{v}_0^i$, i.e.

$$\mathbf{v}_0^i = \mathbf{v}_0^i(x, y, t). \tag{4.16}$$

In order to obtain the asymptotic expansion that is uniform in time, we require that functions $\mathbf{v}_k^i$ $(k = 0, 1, \ldots)$ are bounded in fast variable τ. Further, we assume that all bounded functions of τ which we are dealing with here can be presented in the form

$$f(\tau) = \bar{f} + \tilde{f}(\tau) \tag{4.17}$$

where $\bar{f}$ is the mean value of $f(\tau)$ defined as

$$\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau)\, d\tau \tag{4.18}$$

and $\tilde{f}(\tau) = f(\tau) - \bar{f}$ is the part of $f(\tau)$ that has zero mean value.

Averaging Eq. (4.10), we obtain

$$\partial_t \mathbf{v}_0^i + (\mathbf{v}_0^i \cdot \nabla)\mathbf{v}_0^i = -\nabla \bar{q}_1^i. \tag{4.19}$$

This, together with the incompressibility condition (4.9), represent the Euler equations governing the dynamics of an inviscid incompressible fluid. Eqs. (4.19) and (4.9) are to be solved subject to the boundary conditions (2.21) and the initial conditions (2.22) and (2.23).

The τ-dependent part of Eq. (4.10) yields

$$\partial_\tau \mathbf{v}_1^i + \nabla \tilde{q}_1^i = 0.$$

Writing $\mathbf{v}_1^i$ as the sum of the averaged and τ-dependent parts, $\mathbf{v}_1^i = \bar{\mathbf{v}}_1^i + \tilde{\mathbf{v}}_1^i$, and taking the τ-dependent part of Eq. (4.11), we obtain

$$\partial_\tau \tilde{\mathbf{v}}_1^i + \nabla \tilde{q}_1^i = 0, \tag{4.20}$$

$$\nabla \cdot \tilde{\mathbf{v}}_1^i = 0. \tag{4.21}$$

Evolution equation for $\bar{\mathbf{v}}_1^i$ can be obtained by averaging Eq. (4.12). Boundary conditions for Eqs. (4.20) and (4.21) will be specified later.

4.2. Boundary layer

As before, we assume that

$$u^b = u_0^b + \nu u_1^b + \cdots, \quad v^b = v_0^b + \nu v_1^b + \cdots, \quad q^b = q_0^b + \nu q_1^b + \cdots. \tag{4.22}$$

Now, in contrast with (2.24), quantities u_k^b, v_k^b and q_k^b depend not only on x, s and t, but also on the fast time τ. We substitute (4.4)–(4.7), (4.22) into Eqs. (4.1)–(4.3) and take into account that u_k^i, v_k^i, q_k^i $(k = 0, 1, \ldots)$ satisfy Eqs. (4.8)–(4.13). Then we make the change of variable $y = \nu s$, expand every function of νs in Taylor's series at $\nu = 0$ and collect terms of the equal powers in ν. In the leading order, we obtain (cf. Eqs. (2.25)–(2.27))

$$\partial_\tau u_0^b + V(x,t)\partial_s u_0^b = -\partial_x q_0^b + \partial_s^2 u_0^b, \tag{4.23}$$

$$0 = -\partial_s q_0^b, \tag{4.24}$$

$$\partial_x u_0^b + \partial_s v_0^b = 0. \tag{4.25}$$

As before, we require that

$$u_0^b \to 0, \quad v_0^b \to 0, \quad q_0^b \to 0 \quad \text{as } s \to \infty. \tag{4.26}$$

The condition of decay at infinity (in the boundary layer variable s) for q_0^b and Eq. (4.24) have a consequence that $q_0^b \equiv 0$. Thus, we have

$$\partial_\tau u_0^b + V(x,t)\partial_s u_0^b = \partial_s^2 u_0^b, \tag{4.27}$$

$$\partial_x u_0^b + \partial_s v_0^b = 0. \tag{4.28}$$

Boundary and initial conditions for u_0^b at $s = 0$ are given by

$$u_0^b(x,s,t,\tau)|_{s=0} = h_0(x,t) \equiv U(x,t) - u_0^i(x,0,t), \tag{4.29}$$

and

$$u_0^b(x,s,t,\tau)|_{t=0,\tau=0} = 0. \tag{4.30}$$

Averaging Eqs. (4.27) and (4.29) yields the equations and

$$V(x,t)\partial_s \bar{u}_0^b = \partial_s^2 \bar{u}_0^b, \tag{4.31}$$

and the boundary condition

$$\bar{u}_0^b(x,s,t)|_{s=0} = h_0(x,t). \tag{4.32}$$

These coincide with Eqs. (2.29) and (2.30), and the solution is (cf. (2.32))

$$\bar{u}_0^b(x,s,t) = h_0(x,t)e^{V(x,t)s}. \tag{4.33}$$

The τ-dependent part of u_0^b must be a solution of the equation

$$\partial_\tau \tilde{u}_0^b + V(x,t)\partial_s \tilde{u}_0^b = \partial_s^2 \tilde{u}_0^b, \tag{4.34}$$

subject to the boundary and initial conditions

$$\tilde{u}_0^b|_{s=0} = 0, \qquad \tilde{u}_0^b \to 0 \quad \text{as } s \to \infty; \tag{4.35}$$

$$\tilde{u}_0^b|_{t=0,\tau=0} = -\bar{u}_0^b(x,s,0) = -h_0(x,0)e^{V(x,0)s}. \tag{4.36}$$

It will be verified *a posteriori* that, in the limit $\nu \to 0$, $\tilde{u}_0^b$ rapidly decays outside a thin 'temporary boundary layer' near $t = 0$. With this in mind, we can write

$$V(x,t) = V(x,\nu\tau) = V(x,0) + \nu\tau V_t(x,0) + \cdots.$$

Then we replace $V(x,t)$ in Eq. (4.34) by $V(x,0)$ and move the other terms in this formula to the higher order terms of the expansion.

Thus, we need to find a solution of the equation

$$\partial_\tau \tilde{u}_0^b + V(x,0)\partial_s \tilde{u}_0^b = \partial_s^2 \tilde{u}_0^b \tag{4.37}$$

which satisfies the boundary and initial conditions (4.35), (4.36).

The substitution

$$u_0^b(x,s,\tau) = e^{-\frac{\bar{V}^2}{4}\tau + \frac{\bar{V}}{2}s}\hat{u}(x,s,\tau), \tag{4.38}$$

where $\bar{V} = V(x,0)$, reduces Eq. (4.37) to the heat equation for $\hat{u}$:

$$\hat{u}_\tau = \hat{u}_{ss}. \tag{4.39}$$

The boundary conditions for $\hat{u}$ are

$$\hat{u}|_{s=0} = 0, \qquad \hat{u} \to 0 \quad \text{as } s \to \infty, \tag{4.40}$$

and the initial condition

$$\hat{u}|_{\tau=0} = -\bar{h}_0 e^{\frac{\bar{V}}{2}s}, \tag{4.41}$$

where $\bar{h}_0 = h_0(x,0)$. The initial boundary value problem (4.39)–(4.41) can be solved using standard techniques (described, e.g., in [20]). The solution can be written in the form

$$\hat{u} = \frac{\bar{h}_0}{2} e^{\frac{\bar{V}^2}{4}\tau}\left[e^{-\frac{\bar{V}}{2}s}\operatorname{erfc}\left(\frac{s - \bar{V}\tau}{\sqrt{4\tau}}\right) - e^{\frac{\bar{V}}{2}s}\operatorname{erfc}\left(-\frac{s + \bar{V}\tau}{\sqrt{4\tau}}\right)\right]$$

where $\operatorname{erfc}(z)$ is the complementary error function:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\xi^2}\,d\xi.$$

Hence, we obtain

$$\tilde{u}_0^b = \frac{\bar{h}_0}{2}\left[\operatorname{erfc}\left(\frac{s - \bar{V}\tau}{\sqrt{4\tau}}\right) - e^{\bar{V}s}\operatorname{erfc}\left(-\frac{s + \bar{V}\tau}{\sqrt{4\tau}}\right)\right]. \tag{4.42}$$

Using known properties of $\operatorname{erfc}(z)$, one can show that for any fixed τ,

$$\tilde{u}_0^b \approx -\bar{h}_0 e^{\bar{V}s} + \bar{h}_0\sqrt{\frac{\tau}{\pi}}\frac{e^{-\frac{s^2}{4\tau}}}{s}\left[e^{\bar{V}s} + 1\right] \quad \text{as } s \to \infty,$$

and for any fixed s,

$$\tilde{u}_0^b \approx \frac{\bar{h}_0}{\sqrt{\pi\tau}}\frac{e^{-\frac{\bar{V}^2}{4}\tau}}{|\bar{V}|}\left[1 - e^{\bar{V}s}\right] \quad \text{as } \tau \to \infty.$$

So, the inconsistency in initial and boundary conditions at the outlet results in a fast relaxation process which is important only within a short initial time interval. If the initial and boundary conditions at the outlet are consistent ($\bar{h}_0 = h_0(x,0) \equiv 0$), then $\tilde{u}_0^b \equiv 0$ and, therefore, $u_0^b = \bar{u}_0^b = h_0 e^{Vs}$, which is consistent with the asymptotic expansion of Section 2.

The normal velocity $v_0^b(x,s,t,\tau)$ is determined from Eq. (4.28):

$$v_0^b(x,s,t,\tau) = \frac{\partial}{\partial x}\int_s^\infty u_0^b(x,s',t,\tau)\,ds'. \tag{4.43}$$

Again, $v_0^b(x,s,t,\tau)$, given by (4.43), rapidly decays as $s \to \infty$, but, in general, does not satisfy the boundary condition $v_0^b(x,s,t,\tau)|_{s=0} = 0$. Since the correction to v_0^i is νv_0^b, $v_0^b|_{s=0}$ gives us the boundary condition for the next approximation of the outer solution:

$$v_1^i(x,y,t,\tau)|_{y=0} = -v_0^b|_{s=0}. \tag{4.44}$$

4.3. Higher order terms

It follows from (4.44) that

$$\bar{v}_1^i(x,y,t)|_{y=0} = -\bar{v}_0^b|_{s=0}, \tag{4.45}$$

$$\tilde{v}_1^i(x,y,t,\tau)|_{y=0} = -\tilde{v}_0^b|_{s=0}. \tag{4.46}$$

Boundary condition (4.46) together with the condition

$$\tilde{v}_1^i(x,y,t,\tau)|_{y=L} = 0 \tag{4.47}$$

are used to find a solution of Eqs. (4.20), (4.21). The solution of the boundary value problem (4.20), (4.21), (4.46), (4.47) can be written as

$$\tilde{\mathbf{v}}_1^i(x,y,t,\tau) = -\nabla\phi, \tag{4.48}$$

where ϕ is the solution of the problem

$$\nabla^2\phi = 0,$$
$$\phi_y|_{y=0} = \tilde{v}_0^b|_{s=0}, \qquad \phi_y|_{y=L} = 0. \tag{4.49}$$

The averaged part $\bar{\mathbf{v}}_1^i(x,y,t)$ of $\mathbf{v}_1^i(x,y,t,\tau)$ is determined from the equations that are obtained by averaging Eqs. (4.12), (4.13) and are subject to the boundary condition (4.45) and the conditions

$$\bar{v}_1^i|_{y=L} = 0, \quad \bar{u}_1^i|_{y=L} = 0, \quad \bar{v}_1^i|_{t=0} = 0, \quad \bar{u}_1^i|_{t=0} = 0. \tag{4.50}$$

Now we observe that, in general, $\tilde{u}_1^i = -\phi_x$ does not agree with the boundary condition $\tilde{u}_1^i|_{y=L} = 0$. In order to satisfy this condition, we need to construct a viscous correction to $\tilde{u}_1^i(x,y,t,\tau)$ at the inlet. Thus, we have obtained a surprising result: although there is only one boundary layer (at the outlet) in the leading order terms of the asymptotic expansion, *a viscous boundary layer at the inlet* appears in the higher order terms.

To construct a boundary layer at the inlet we assume that (cf. (4.4)–(4.6))

$$u = u^i(x,y,t,\tau) + u^b(x,s,t,\tau) + u^a(x,\xi,t,\tau),$$
$$v = v^i(x,y,t,\tau) + \nu v^b(x,s,t,\tau) + \nu v^a(x,\xi,t,\tau),$$
$$q = q^i(x,y,t,\tau) + q^b(x,s,t,\tau) + q^a(x,\xi,t,\tau), \tag{4.51}$$

where $\xi = (L-y)/\nu$ is the boundary layer variable at the inlet. The same procedure as before leads to the following equations.

In the leading order, we obtain

$$\partial_\tau u_0^a - V(x,t)\partial_\xi u_0^a = \partial_\xi^2 u_0^a. \tag{4.52}$$

Boundary conditions:

$$u_0^a \to 0 \quad \text{as } \xi \to \infty, \qquad u_0^a(x,\xi,t,\tau)|_{\xi=0} = 0. \tag{4.53}$$

The initial condition:

$$u_0^a(x,\xi,t,\tau)|_{\tau=0,t=0} = 0. \tag{4.54}$$

The only solution of Eq. (4.52) satisfying (4.53) and (4.54) is zero solution. Thus, as expected, we obtain

$$u_0^a(x,\xi,t,\tau) \equiv 0, \tag{4.55}$$

and, as a consequence of (4.55), $v_0^a(x,\xi,t,\tau) \equiv 0$. The next order terms are

$$\partial_\tau u_1^a - V(x,t)\partial_\xi u_1^a - \partial_\xi^2 u_1^a = 0. \tag{4.56}$$

(Here we have used Eq. (4.55).) Boundary conditions:

$$u_1^a \to 0 \quad \text{as } \xi \to \infty, \qquad u_1^a|_{\xi=0} = -\tilde{u}_1^i(x,y,t,\tau)|_{y=L}. \tag{4.57}$$

The initial condition:

$$u_1^a|_{t=0,\tau=0} = 0. \tag{4.58}$$

The initial boundary value problem (4.56)–(4.58) can be solved analytically in the same way as problem (4.27), (4.29), (4.30) was solved. Then, we find $v_1^a(x,\xi,t,\tau)$ from the incompressibility equation and use it as the boundary condition at the inlet for the next approximation of the outer solution.

For the boundary layer at the outlet, the first-order term u_1^b is determined from the equation

$$\partial_\tau u_1^b + V(x,t)\partial_s u_1^b = \partial_s^2 u_1^b + a_1(x,s,t,\tau) - \tau V(x,0)\partial_s \tilde{u}_0^b \tag{4.59}$$

where $a_1(x,s,t,\tau)$ is given by Eq. (2.37), and the last term on the hand right side represent the $\mathrm{O}(\nu)$ term in the expansion of $V(x,\nu\tau)$. Boundary and initial conditions for u_1^b are

$$u_1^b \to 0 \quad \text{as } s \to \infty, \qquad u_1^b|_{s=0} = -u_1^i(x,y,t,\tau)|_{y=0}; \tag{4.60}$$

$$u_1^b|_{t=0,\tau=0} = 0. \tag{4.61}$$

When we know the solution of Eqs. (4.59)–(4.61), we employ the incompressibility condition to find v_1^b which, in turn, is used to obtain the boundary conditions at the outlet for the next order terms in the outer expansion.

The procedure described above can be repeated as many times as necessary to give us higher order terms of the asymptotic expansion.

4.4. Example 3

Consider the following two-dimensional viscous flow between two infinite parallel planes which are permeable for the fluid. For $t < 0$, there is a steady flow of the fluid with constant (both in time and in space) velocity with components U and V in the parallel and normal directions respectively. This steady flow can be produced by motion of both planes with constant velocity U and suction and blowing of the fluid through the planes with constant normal velocity V. At $t = 0$ the plane, at which the suction is applied, suddenly comes to a stop, so that it has zero velocity for $t > 0$. The problem is to find the velocity of the fluid for $t > 0$. To describe this flow in the limit of vanishing viscosity, we consider the two-dimensional Navier–Stokes equations (2.1)–(2.3) with the following boundary conditions

$$v|_{y=0} = v|_{y=L} = V, \tag{4.62}$$

$$u|_{y=0} = 0, \qquad u|_{y=L} = U, \tag{4.63}$$

where U and V are constants and $V < 0$. The initial conditions for u and v are

$$v|_{t=0} = V, \qquad u|_{t=0} = U. \tag{4.64}$$

The initial condition for the tangent velocity u is consistent with the boundary condition at $y = L$ (the inlet), but inconsistent with the boundary condition at $y = 0$ (the outlet): $(u(x,y,t)|_{t=0})|_{y=0} \neq 0$. The leading term of the outer expansion is given by

$$v_0^i = V, \qquad u_0^i = U.$$

The leading term of the boundary layer part of the expansion is given by Eqs. (4.33) and (4.42) with

$$h_0(x,t) = -u_0^i|_{y=0} = -U, \quad \bar{h}_0 = -U \quad \text{and} \quad \bar{V} = V.$$

Hence, we obtain

$$\begin{aligned} u &= u_0^i + u_0^b + \mathrm{O}(\nu) \\ &= U(1 - e^{\frac{V}{\nu}y}) - \frac{U}{2}\left[\operatorname{erfc}\left(\frac{y - Vt}{\sqrt{4\nu t}}\right) - e^{\frac{V}{\nu}y}\operatorname{erfc}\left(-\frac{y + Vt}{\sqrt{4\nu t}}\right)\right]. \end{aligned} \tag{4.65}$$

Note that, as $\tau \to \infty$, $u(y,\tau)$ tends to the asymptotic suction profile (see, e.g., [13]):

$$u \to U(1 - e^{\frac{V}{\nu} y}) + \mathrm{O}(\nu).$$

Problem (2.1)–(2.3), (4.62)–(4.64) can be solved analytically. To do this, we first observe that the boundary and initial conditions for $v(x, y, t)$ imply that $v(x, y, t) = V$. This and the incompressibility condition (2.3) have a consequence that $u = u(y, t)$. Then it follows from Eq. (2.2) that $p = p(x, t)$. Eq. (2.1) reduces to

$$u_t + V u_y = -p_x + \nu u_{yy}.$$

Finally, the fact that u is independent of x and the requirement that p is bounded for all x imply that $p(x, t) = C(t)$ for some $C(t)$, so that $p_x = 0$, and we obtain

$$u_t + V u_y = \nu u_{yy}. \tag{4.66}$$

It is convenient to present $u(y, t)$ in the form

$$u(y, t) = U \frac{1 - e^{\frac{V}{\nu} y}}{1 - e^{\frac{V}{\nu} L}} + g(y, t).$$

Here the first term represents the steady solution of Eq. (4.66) that satisfies the boundary conditions (4.63), and $g(y, t)$ is the solution of the problem

$$\begin{aligned} &g_t + V g_y = \nu g_{yy}, \\ &g(y, t)|_{y=0} = g(y, t)|_{y=L} = 0, \\ &g(y, t)|_{t=0} = \frac{U}{1 - e^{\frac{V}{\nu} L}} (e^{\frac{V}{\nu} y} - e^{\frac{V}{\nu} L}). \end{aligned} \tag{4.67}$$

Substituting $g = \exp(-\frac{V^2}{4\nu} t + \frac{V}{2\nu} y) \hat{g}(y, t)$ in the equation for g, we reduce it to the heat equation for $\hat{g}$

$$\hat{g}_t = \nu \hat{g}_{yy}, \tag{4.68}$$

which is to be solved subject to the following boundary and initial conditions

$$\hat{g}|_{y=0} = \hat{g}|_{y=L} = 0, \qquad \hat{g}|_{t=0} = \frac{U e^{-\frac{V}{2\nu} y}}{1 - e^{\frac{V}{\nu} L}} (e^{\frac{V}{\nu} y} - e^{\frac{V}{\nu} L}). \tag{4.69}$$

The solution, obtained by standard methods [20], is given by

$$\hat{g} = \sum_{n=1}^{\infty} C_n e^{-(\frac{\pi n}{L})^2 \nu t} \sin\left(\frac{\pi n}{L} y\right), \quad C_n = \frac{2\pi n U}{(\frac{VL}{2\nu})^2 + (\pi n)^2}.$$

Finally, recalling the definition of $\hat{g}$, we obtain

$$u(y, t) = U \frac{1 - e^{\frac{V}{\nu} y}}{1 - e^{\frac{V}{\nu} L}} + e^{-\frac{V^2}{4\nu} t + \frac{V}{2\nu} y} \sum_{n=1}^{\infty} C_n e^{-(\frac{\pi n}{L})^2 \nu t} \sin\left(\frac{\pi n}{L} y\right). \tag{4.70}$$

The asymptotic and exact solutions, given by Eqs. (4.65) and (4.70) respectively, are shown in Fig. 5.

4.5. Example 4

Consider now the same particular problem as in Example 1 (Section 2.5) but with the different boundary condition for u at the outlet (cf. Eq. (2.59)):

$$u|_{y=0} = \sin t. \tag{4.71}$$

This boundary condition satisfies the primary consistency condition given by the first equation (2.11) but does not satisfy the first-order consistency condition (2.57).

In this case, the outer asymptotic expansion is the same as in Example 1, so that the first two terms of the outer expansion are given by Eqs. (2.64) and (2.65). The leading term of the boundary layer part of the expansion is $u_0^b = h_0(t) e^{-s}$ where

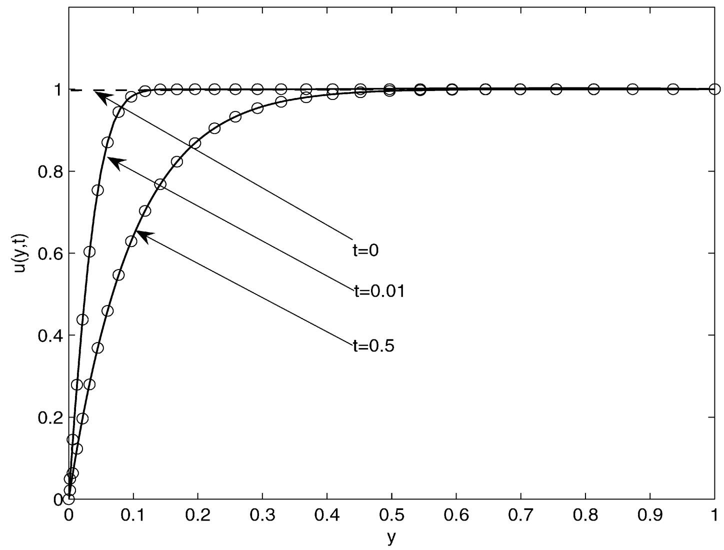

Fig. 5. Asymptotic and exacts solutions $u(y,t)$ at $t=0.01$ and $t=0.5$ for $\nu=0.1$, $L=U=1$ and $V=-1$. Solid curves correspond to the exact solution (4.70), the circles represent the asymptotic solution given by Eq. (4.65), and the dotted line is the initial velocity $u(y,t)|_{t=0}$.

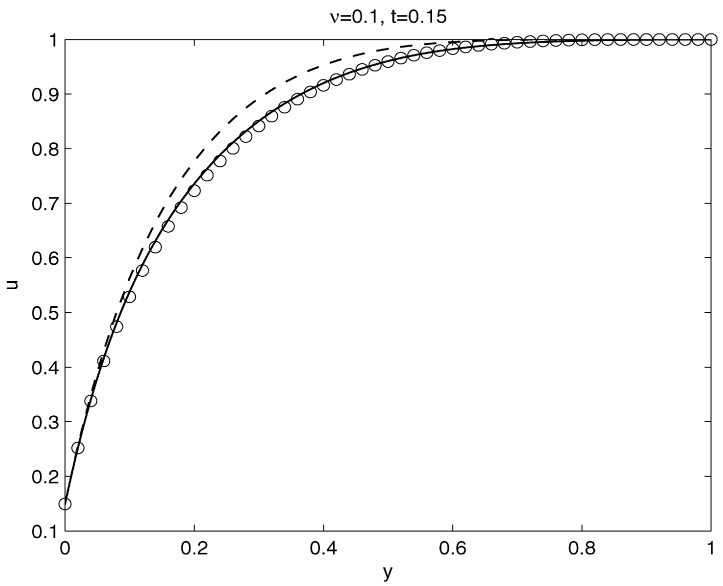

Fig. 6. Leading order and first-order asymptotic solutions $u(y,t)|_{t=0.15}$ for $\nu=0.1$. Dashed line corresponds to the leading order approximation $u_0=u_0^i+u_0^b$, solid line – to the numerical solution, and circles represent the first-order asymptotic solution $u_0+\nu u_1$ where $u_1=u_1^i+u_1^b$.

$$h_0=\begin{cases}\sin t-1+(1-t)^4, & 0\leqslant t\leqslant 1,\\ \sin t-1, & t>1.\end{cases}$$

Since the first-order consistency condition is not satisfied, the first-order term of the boundary layer part of the expansion must be determined by solving problem (4.59)–(4.61). It can be shown that it is given by

$$u_1^b=\big[h_1(t)-sh_{0t}(t)\big]e^{-s}+\bar{h}_{0t}(s-\tau)e^{-s}+\frac{\bar{h}_{0t}}{2}\bigg[(s+\tau)\,\mathrm{erfc}\bigg(\frac{s+\tau}{\sqrt{4\tau}}\bigg)-e^{-s}(\tau-s)\,\mathrm{erfc}\bigg(\frac{s-\tau}{\sqrt{4\tau}}\bigg)\bigg]. \quad (4.72)$$

Here $\bar{h}_{0t} = h_{0t}(0)$ and

$$h_1 = \begin{cases} 12t(1-t)^2, & 0 \leqslant t \leqslant 1, \\ 0, & t > 1. \end{cases} \tag{4.73}$$

The leading order asymptotic solution $u_0 = u_0^i + u_0^b$, the first-order asymptotic solution $u_0 + \nu u_1$, where $u_1 = u_1^i + u_1^b$, and the numerical solution u, obtained with the help of the MATLAB PDE solver, are shown in Fig. 6.

5. Conclusions

We have constructed a formal asymptotic expansion of solutions to the Navier–Stokes equations in the limit of vanishing viscosity (high Reynolds number) in the case of flows through a given domain when the fluid flows into the domain through one part of the boundary (the inlet) and leaves the domain through the other part (the outlet). In the leading order the velocity is the sum of two terms. The first term is a solution of the Euler equations and describes the flow everywhere in the domain except for a thin boundary layer near the outlet. The second term represents a boundary layer correction to the inviscid solution. The second term is a solution of boundary layer equations which, in contrast with Prandtl's boundary layer, are linear and can be solved analytically in general case. An interesting feature of the boundary layer equations is that the corresponding initial boundary value problem is over-determined, but nevertheless has a unique solution provided that the initial and boundary conditions for the velocity in the original problem are consistent. We have also computed the $O(\nu)$ terms of the expansion and found that in order to satisfy the initial condition, we have to impose an additional consistency requirement on the initial and boundary conditions for the tangent velocity at the outlet. It has been shown that the asymptotic expansion can be continued up to terms of arbitrary order in ν, and each successive approximation requires a new consistency condition. Also, we have constructed an asymptotic expansion in the case of inconsistent initial and boundary conditions. Such mathematical problems can be used to describe real fluid flows produced by a sudden change in the tangent velocity at the outlet. In this case, the asymptotic solution includes a rapidly decaying component which describes the relaxation of the initial discontinuity. It is interesting that, in general, the inconsistency in initial and boundary conditions at the outlet results in the appearance of a boundary layer at the inlet, which is in sharp contrast with the case of consistent initial and boundary conditions where there is no boundary layer at the inlet in all orders of the asymptotic expansion. In both cases, the examples presented in the paper show that asymptotic solutions are in quite good agreement with exact or numerical solution

In the case of consistent initial and boundary conditions, the procedure of constructing the boundary layer at the outlet does not depend on the boundary conditions at the inlet. This means that our asymptotic expansion can be easily adapted to the situations where different boundary conditions at the inlet are employed. For example, one can consider flows through a given domain with the normal velocity and the tangent vorticity rather than the tangent velocity prescribed at the inlet. Inviscid flows of this type were studied in [1,2,7,8]. If all components of the velocity are specified at the outlet, the asymptotic expansion constructed here shows that the effect of small viscosity on such flows is negligible everywhere in the flow domain except for a thin boundary layer at the outlet.

Much remains to be done in this area. In particular, it would be interesting to investigate the effect of small viscosity on the stability of steady flows through a given domain. For example, it is known that certain inviscid flows with a given tangent vorticity at the inlet can be asymptotically stable (see [7]). On the other hand, the example of the asymptotic suction profile shows that a boundary layer at the outlet can be unstable [21,22]. In this context, an interesting open question is whether viscosity can destabilise flows that are asymptotically stable in the framework of the inviscid theory. This is a problem for a further investigation.

Another open problem is to construct an asymptotic expansion in the case when initial and boundary conditions for the tangent velocity *at the inlet* are inconsistent. It is known that in this case, in inviscid approximation, there is a moving front at which the velocity (and/or its derivatives) is discontinuous [1,2,9]. The presence of viscosity, however small, will smoothen the discontinuity, but mathematical description of this phenomenon is absent.

In this paper, we studied only flows though a domain whose whole boundary is permeable. The most important open problem is to construct an asymptotic expansion for nearly inviscid flows though a domain whose boundary consists of both permeable and impermeable parts, such as a flow though a channel of finite length. Such analysis would inevitably include an investigation of the interaction between the Prandtl boundary layer at an impermeable wall and the boundary layer at the outlet constructed here. This is also a problem for a further study.

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